**Problem 1.** Find

$$\int x^5\sqrt{1+x^2}\,dx$$

*Solution.* Our intention is to get rid of the radical, so we apply the change of variables

$$u = 1 + x^2$$

which means that $du = 2x\,dx$ or equivalently

$$x\,dx = \frac{1}{2}du$$

Now we have to express $x$ as function of the new variable $u$. We obtain $x^2 = u-1$, which implies $x^4 = (u-1)^2$. We use all this to solve our integral in the following way:

$$\begin{aligned}
\int x^5\sqrt{1+x^2}\,dx &= \int x^4\sqrt{1+x^2}x\,dx \\
&= \int \sqrt{u}(u-1)^2\,\frac{du}{2} \\
&= \frac{1}{2}\int \sqrt{u}(u^2-2u+1)\,du \\
&= \frac{1}{2}\int (u^{5/2}-2u^{3/2}+u^{1/2})\,du \\
&= \frac{1}{2}\left(\frac{2}{7}u^{7/2}-2\cdot\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+C \\
&= \frac{1}{7}(1+x^2)^{7/2}-\frac{2}{5}(1+x^2)^{5/2}+\frac{1}{3}(1+x^2)^{3/2}+C
\end{aligned}$$

which is the final answer. □